→ Fuzzy relation R is a mapping from cartesian space $X \times Y$ to the interval $[0,1]$ where the strength of mapping is expressed by membership function $\mu_{\tilde{R}}(x,y)$

CORDINALITY OF FUZZY RELATIONS :

→ The cordinality (grouping) of fuzzy relation b/w two or more universe is infinity

→ OPERATIONS ON FUZZY RELATIONS :

→ let $\tilde{R}$ & $\tilde{S}$ be the fuzzy relations on the cartesian space $X \times Y$

1. UNION OPERATION :

→ $$\mu_{\underset{\sim}{R} \cup \underset{\sim}{S}}(x,y) = \max\left(\mu_{\underset{\sim}{R}}(x,y), \mu_{\underset{\sim}{S}}(x,y)\right)$$

2. INTERSECTION OPERATION :

$$\mu_{\underset{\sim}{R} \cap \underset{\sim}{S}}(x,y) = \min\left(\mu_{\underset{\sim}{R}}(x,y), \mu_{\underset{\sim}{S}}(x,y)\right)$$

3. COMPLIMENT OPERATION :

$$\mu_{\bar{\underset{\sim}{R}}}(x,y) = 1 - \mu_{\underset{\sim}{R}}(x,y)$$

4. CONTAINMENT OPERATION :

$$\underset{\sim}{R} \subset \underset{\sim}{S} \overset{\text{only if}}{\Rightarrow} \mu_{\underset{\sim}{R}}(x,y) \leq \mu_{\underset{\sim}{S}}(x,y)$$

PROPERTIES OF FUZZY RELATION :

→ The properties of computativity, associativity, distributivity & idempotency hold good for fuzzy relation.

→ De-morgan's laws also holds good for fuzzy relation. But excluded middle law does not hold good for fuzzy relation because their is a overlap b/w relation & its compliment.

- In fuzzy relation the symbols for X, ϕ are replaced by E, O respectively.

- Therefore $\underset{\sim}{R} \cup \overline{\underset{\sim}{R}} \neq E$ &

$$\underset{\sim}{R} \cap \overline{\underset{\sim}{R}} \neq O$$

FUZZY CARTESIAN PRODUCT & COMPOSITION :

F.C.P :

→ Let $\underset{\sim}{A}$ be the fuzzy set on universe X & $\underset{\sim}{B}$ be a fuzzy set on universe Y, then the cartesian product b/w $\underset{\sim}{A}$ & $\underset{\sim}{B}$ results in a fuzzy relation $\underset{\sim}{R}$ which is contained within full cartesian product space i.e

which is contain in

$$\underset{\sim}{A} \times \underset{\sim}{B} = \underset{\sim}{R} \subset X \times Y$$

where fuzzy relation $\underset{\sim}{R}$ has membership funct given by

$$\mu_{\underset{\sim}{R}}(x,y) = \mu_{\underset{\sim}{A} \times \underset{\sim}{B}}(x,y) = \min\left(\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(y)\right)$$

Fuzzy Composition :

- Let $\underset{\sim}{R}$ is a fuzzy relation on cartesian space

$\underset{\sim}{X} \times \underset{\sim}{Y}$, $\underset{\sim}{S}$ is a fuzzy relation on cartesian space $\underset{\sim}{Y} \times \underset{\sim}{Z}$, $\underset{\sim}{T}$ is a fuzzy relation on $\underset{\sim}{X} \times \underset{\sim}{Z}$ then

$\underset{\sim}{R} \rightarrow \underset{\sim}{X} \times \underset{\sim}{Y}$

$\underset{\sim}{S} \rightarrow \underset{\sim}{Y} \times \underset{\sim}{Z}$

$\underset{\sim}{T} \rightarrow \underset{\sim}{X} \times \underset{\sim}{Z}$ then Fuzzy max-min composition is defined in terms of set-theoretic notation and membership function theoretic notation in following way

o → fuzzy composition

$$\underset{\sim}{T} = \underset{\sim}{R} \circ \underset{\sim}{S}$$

- Its membership funct is expressed as

$$\mu_{\underset{\sim}{T}}(x, z) = \overset{max}{\bigvee_{y \in Y}} \left(\mu_{\underset{\sim}{R}}(x, y) \overset{min}{\wedge} \mu_{\underset{\sim}{S}}(y, z) \right)$$

max - product composition:

→ max - product composition is defined in terms of membership function as

$$\mu_{\underset{\sim}{T}}(x, z) = \bigvee_{y \in Y} \left(\mu_{\underset{\sim}{R}}(x, y) \cdot \mu_{\underset{\sim}{S}}(y, z) \right) \quad \&$$

$\boxed{R \circ S \neq S \circ R}$ → property of fuzzy relation.

FUZZY TOLERANCE OR FUZZY RESEMBLENCE RELATION

- It is a fuzzy tolerance or resemblence relation if following two properties of matrix relations holds good they are

1. Reflexivity :

$$\mu_{\underset{\sim}{R}}(x_i, x_i) = 1$$

2. Symmetry :

$$\mu_{\underset{\sim}{R}}(x_i, x_j) = \mu_{\underset{\sim}{R}}(x_j, x_i)$$

FUZZY EQUIVALENCE OR SIMILARITY RELATION :

- The properties of reflexivity & symmetry holds good

i.e $\mu_{\underset{\sim}{R}}(x_i, x_i) = 1$

$$\mu_{\underset{\sim}{R}}(x_i, x_j) = \mu_{\underset{\sim}{R}}(x_j, x_i)$$

One more property should also satisfy

3. Transitivity :

$$\mu_{\underset{\sim}{R}}(x_i, x_j) = \lambda_1 \ \&$$

$$\mu_{\underset{\sim}{R}}(x_j, x_K) = \lambda_2 \text{ then relation}$$

$$\mu_{\underset{\sim}{R}}(x_j, x_K) = \lambda \text{ where}$$

$$\lambda \geq \min[\lambda_1, \lambda_2]$$

PROBLEM-

(General) representation

- Let $X = \{x_1, x_2\} = \{(NYC, TKO)\}$

$Y = \{y_1, y_2, y_3\} = \{TPE, HKG, BJI\}$ if R is a crisp relation then it is defined by following characteristic funct $\mu_R(x_i, y_j)$

		y_1	y_2	y_3
x_1	NYC	TPE	HKG	BJI
x_2	TKO			

		y_1 TPE	y_2 HKG	y_3 BJI
x_1	NYC	0	0	0
x_2	TKO	1	1	1

if R is considered as fuzzy relation then it is defined by following membership funct

		y_1 TPE	y_2 HKG	y_3 BJI
x_1	NYC	0.3	0.1	0.1
x_2	TKO	1	0.7	0.8

$$\underset{\sim}{R}(X,Y) = \frac{0.3}{(NYC, TPE)} + \frac{0.1}{(NYC, HKG)} + \frac{0.1}{(NYC, BJI)} +$$

$$\frac{1}{(TKO, \cancel{HKG}\ TPE)} + \frac{0.7}{(TKO, HKG)} + \frac{0.8}{(TKO, BJI)}$$

(Fuzzy relation)

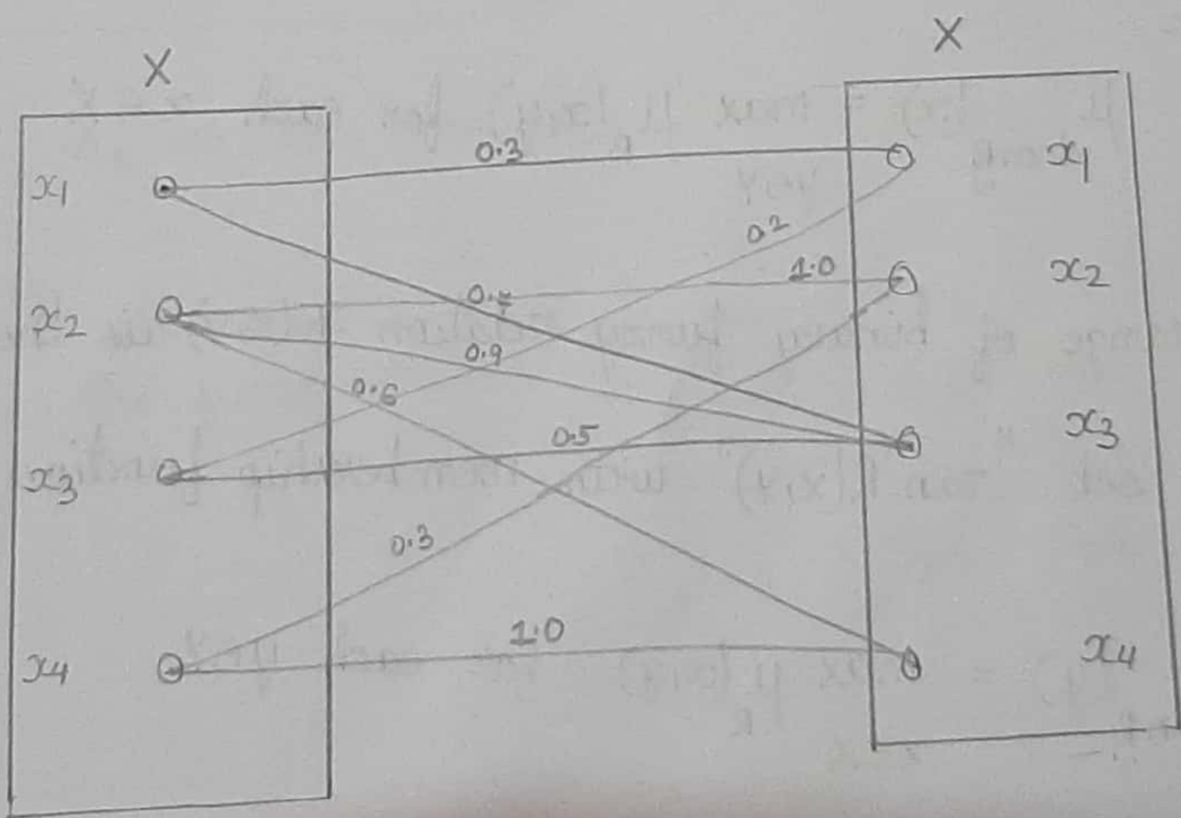
→ Let $X = \{x_1, x_2, x_3, x_4\}$ consider the following binary fuzzy relation on X

$$R(X,X) = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{bmatrix} 0.3 & 0 & 0.7 & 0 \\ 0 & 1 & 0.9 & 0.6 \\ 0.2 & 0 & 0.5 & 0 \\ 0 & 0.3 & 0 & 1 \end{bmatrix} \end{array}$$

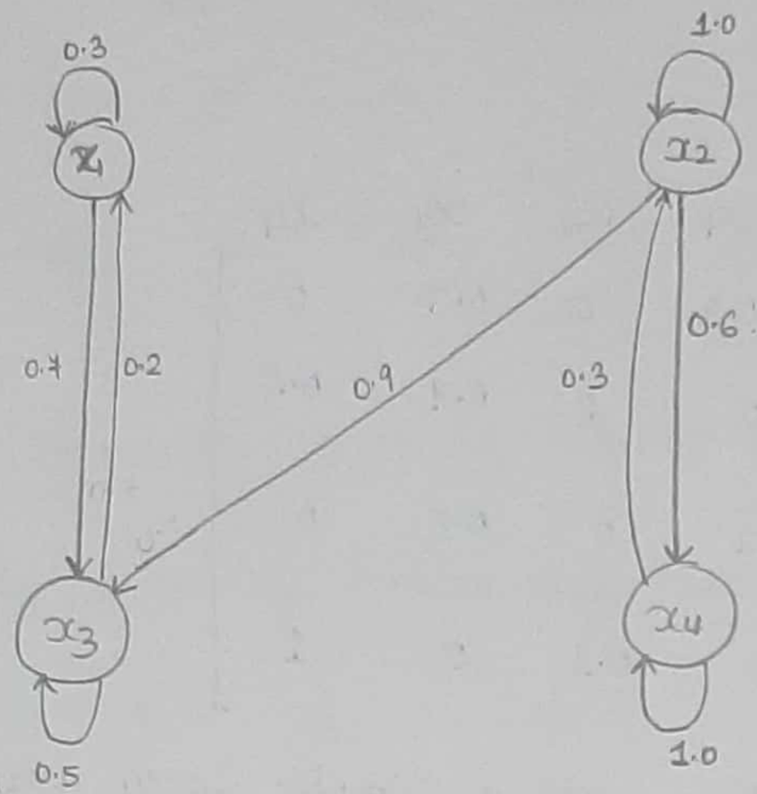
having mapping

Draw the bipartite graph & simple fuzzy graph of the relation $R(X, X)$

a) BIPARTITE GRAPH :



b) SIMPLE FUZZY GRAPH - (arrow imp)



Property :

- The domain of the binary/normal fuzzy relation $R(x,y)$ is the fuzzy set "dom $R(x,y)$" with membership function

$$\mu_{dom \tilde{R}}(x) = \max_{y \in Y} \mu_R(x,y) \text{ for each } x \in X$$

The range of binary fuzzy relation $R(x,y)$ is the fuzzy set "ran $R(x,y)$" with membership function

$$\mu_{ran \tilde{R}}(y) = \max_{x \in X} \mu_R(x,y) \text{ for each } y \in Y$$

- The height of fuzzy relation $\underset{\sim}{R}$ is a number $H(R)$ defined by

$$H(R) = \overset{\text{support}}{\underset{y \in Y}{\sup}} \; \underset{x \in X}{\sup} \; \mu_{\underset{\sim}{R}}(x,y)$$

- If $H(R) = 1$ then '$\underset{\sim}{R}$' is a normal fuzzy relation else it is a subnormal fuzzy relation

- The resolution principle of fuzzy sets also holds good for every binary fuzzy relation $R(X,Y)$ which can be represented in its resolution form as

$$\boxed{R = \overset{\text{for all}}{\bigcup_{\alpha \in \Lambda_R}} \alpha R_\alpha}$$

Resolution property

$\Lambda_R \to$ level set

where Λ_R is the level set

- PROBLEM-

$\Rightarrow$(Resolution)

$\Rightarrow$ Consider the fuzzy relation $R = \begin{bmatrix} 0.4 & 0.5 & 0 \\ 0.9 & 0.5 & 0 \\ 0 & 0 & 0.3 \\ 0.3 & 0.9 & 0.4 \end{bmatrix}$

&

$$R = 0.3\,R_{0.3} + 0.4\,R_{0.4} + 0.5\,R_{0.5} + 0.9\,R_{0.9}$$

(resolution)

$0.4 > 0.3$

$$R = 0.3\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} + 0.4\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$+ 0.5\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + 0.9\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Other Operations on Fuzzy Relations:

Projection of R

1. Given fuzzy relation $R(X, Y)$, let $[R \downarrow Y]$ denote the projection of R onto Y. Then

$[R \downarrow Y]$ is a fuzzy relation or set in Y whose membership function denoted by

$$\mu_{[R \downarrow Y]}(y) = \max_{x} \mu_R(x, y)$$

row

$x \to$ row, $y \to$ column

- Consider fuzzy relation

$$R(X,Y) = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccccc} y_1 & y_2 & y_3 & y_4 & y_5 \end{array} \\ \begin{bmatrix} 0.2 & 0.5 & 1 & 0 & 0.6 \\ 0.1 & 0 & 0.7 & 0.4 & 0 \\ 0.9 & 0.2 & 0 & 0.2 & 1 \end{bmatrix} \end{array}$$

then $[R \downarrow Y]$

max in each row

$$[R \downarrow Y] = \frac{1}{x_1} + \frac{0.7}{x_2} + \frac{1.0}{x_3}$$

max in each column

$$[R \downarrow \underline{X}] = \frac{0.9}{y_1} + \frac{0.5}{y_2} + \frac{1}{y_3} + \frac{0.4}{y_4} + \frac{1}{y_5}$$

row

opposite

3 set relation

2. Consider a 3-ary (ternary) fuzzy relation

Q) Let $X_1 = \{a, b\}$

$X_2 = \{c, d\}$ & $X_3 = \{f, g\}$

relation b/w 3 sets represented by

$$R(X_1, X_2, X_3) = \frac{0.3}{acf} + \frac{0.7}{adf} + \frac{0.1}{bcg} + \frac{0.8}{bdf} + \frac{1}{bdg}$$

then

Sol:

$$R_{1,2} \equiv \{R \downarrow \{X_1, X_2\}]$$

$$R_{1,2} \equiv \left[R \downarrow \{X_1, X_2\}\right] = \frac{0.3}{ac} + \frac{0.7}{ad} + \frac{0.1}{bc} + \underbrace{\frac{0.8}{bd} + \frac{1}{bd}}_{max}$$

$$= \frac{0.3}{ac} + \frac{0.7}{ad} + \frac{0.1}{bc} + \frac{1}{bd}$$

$$R_{1,3} \equiv \left[R \downarrow \{X_1, X_3\}\right] = \frac{0.3}{af} + \frac{0.7}{af} + \frac{0.1}{bg} + \frac{1}{bg} + \frac{0.8}{bf}$$

$$= \frac{0.7}{af} + \frac{1}{bg} + \frac{0.8}{bf}$$

$$R_{2,3} \equiv \left[R \downarrow \{X_2, X_3\}\right] = \frac{0.3}{cf} + \frac{0.8}{df} + \frac{0.1}{cg} + \frac{1}{dg}$$

$$R_1 \equiv \left[R \downarrow X_1\right] = \frac{0.7}{a} + \frac{1}{b}$$

$$R_2 \equiv \left[R \downarrow X_2\right] = \frac{0.3}{c} + \frac{1}{d}$$

$$R_3 \equiv \left[R \downarrow X_3\right] = \frac{0.8}{f} + \frac{1}{g}$$

II **Cylindrical Extension:**

Given a fuzzy relation $R(X)$ (or) a fuzzy set R on X. Let $[R \uparrow Y]$ denote the cylindrical extension of R into Y

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then $[R \uparrow Y]$ is a fuzzy relation in $X \times Y$ with a membership function defined by

$$\mu_{[R \uparrow Y]}(x,y) \equiv \mu_R(x) \text{ for every } x \in X, y \in Y$$

→ We use the cartesian product to define cylindrical extension

- Consider $X \times Y$ is the complete cartesian product space. Let R be a fuzzy set on X where $X = \{x_1, x_2, --- x_n\}$. The cylindrical extension of R into Y where $Y = \{y_1, y_2, --- y_m\}$ can be obtained by $[R \uparrow Y]$

$$[R \uparrow Y] = R \times Y$$

where we consider $Y = \frac{1}{y_1} + \frac{1}{y_2} + ---- + \frac{1}{y_m}$. similarly if R is a fuzzy set on Y then

$$[R \uparrow X] = X \times R$$

→ Consider $X = \{x_1, x_2\}$ & $Y = \{y_1, y_2\}$

$$R = \frac{\mu_R(x_1)}{x_1} + \frac{\mu_R(x_2)}{x_2} \text{ \& } Y = \frac{1}{y_1} + \frac{1}{y_2} \text{ then}$$

$$[R \uparrow Y] = R \times Y$$

$$[R\uparrow Y] = R \times Y$$

$$= \frac{\mu_R(x_1)}{(x_1, y_1)} + \frac{\mu_R(x_1)}{(x_1, y_2)} + \frac{\mu_R(x_2)}{(x_2, y_1)} + \frac{\mu_R(x_2)}{(x_2, y_2)}$$

Problem

$$\rightarrow \quad R(X, Y) = \begin{array}{c c} & \begin{array}{ccccc} y_1 & y_2 & y_3 & y_4 & y_5 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} 0.2 & 0.5 & 1 & 0 & 0.6 \\ 0.1 & 0 & 0.7 & 0.4 & 0 \\ 0.9 & 0.2 & 0 & 0.2 & 1 \end{bmatrix} \end{array}$$

$$[R\uparrow Y] = \frac{0.2}{(x_1, y_1)} + \frac{0.5}{(x_1, y_2)} + \frac{1}{(x_1, y_3)} + \cancel{0} + \frac{0.6}{(x_1, y_5)}$$

$$+ \frac{0.1}{(x_2, y_1)} + \frac{0.7}{(x_2, y_3)} + \frac{0.4}{(x_2, y_4)} + \frac{0.9}{(x_3, y_1)} + \frac{0.2}{(x_3, y_2)}$$

$$+ \frac{0.2}{(x_3, y_4)} + \frac{1}{(x_3, y_5)}$$

$\frac{0.9}{y_1}$

$$\Rightarrow \quad \left[[R\downarrow X] \uparrow Y\right]$$

$$[R\downarrow X] = \left[\frac{0.9}{y_1} + \frac{0.5}{y_2} + \frac{1}{y_3} + \frac{0.4}{y_4} + \frac{1}{y_5} \right]$$

$$\left[[R \downarrow X] \uparrow Y\right] = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0.7 & 0.7 & 0.7 & 0.7 & 0.7 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$ max element in row.

$$\left[R \downarrow Y] \uparrow X\right] = \begin{bmatrix} 0.9 & 0.5 & 1 & 0.4 & 1 \\ 0.9 & 0.5 & 1 & 0.4 & 1 \\ 0.9 & 0.5 & 1 & 0.4 & 1 \end{bmatrix}$$

cylindrical extension is done b/w diff sets.

CYLINDRICAL CLOSURE :

- Let $Y = Y_1 \times Y_2 \times \cdots Y_n$ be the complete cartesian product space where Y_i is more than 1-dimension

Given a set of projection of fuzzy relation, the cylindrical closure of these projections denoted by

$I \to$ cylindrical closure

$C_y \to$ cylinder

$C_y I\{R_i\}$ is defined by

$$C_y I\{R_i\} = \left[R_1 \uparrow (Y - Y_1)\right] \cap \left[R_2 \uparrow (Y - Y_2)\right] \cap \cdots \cdots \cap \left[R_n \uparrow (Y - Y_n)\right]$$

where $Y - Y_i$ is the cartesian product space without Y_i (i.e. $Y_1 \times Y_2 \times \cdots Y_{i-1} \times Y_{i+1} \times \cdots Y_n$) & the intersection symbol is a t-norm operator

→ The cylindrical closures of

$R_{1,2}$, $R_{1,3}$, $R_{2,3}$ is given by

$$Cyl\{R_{1,2}, R_{1,3}, R_{2,3}\} = [R_{1,2} \uparrow X_3] \cap [R_{1,3} \uparrow X_2] \cap [R_{2,3} \uparrow X_1]$$

$$= \frac{0.3}{acf} + \frac{0.4}{adf} + \frac{0.1}{bcf} + \frac{0.1}{bcg} + \frac{0.8}{bdf} + \frac{1}{bdg}$$

(From textbook table) (will not come in exam)

which is very close to original relation $R(x_1, x_2, x_3)$

→ Let $R_X \triangleq [R \downarrow X]$ & $R_Y \triangleq [R \downarrow Y]$ then the

cylindrical closure of R_X, R_Y

$$Cyl\{R_X, R_Y\} = [R_X \uparrow Y] \cap [R_Y \uparrow X]$$

$$= [R_X \times Y] \cap [X \times R_Y]$$

$$= R_X \times R_Y$$

$$\therefore \quad R(X, Y) \subseteq R_X \times R_Y$$

(subset or contentment property)

$$R(x,y) = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccccc} y_1 & y_2 & y_3 & y_4 & y_5 \end{array} \\ \begin{bmatrix} 0.2 & 0.5 & 1 & 0 & 0.6 \\ 0.1 & 0 & 0.7 & 0.4 & 0 \\ 0.9 & 0.2 & 0 & 0.2 & 1 \end{bmatrix} \end{array}$$

$$R_X \times R_Y = \Big[[R \downarrow X] \uparrow Y \Big] \cap \Big[[R \downarrow Y] \uparrow X \Big]$$

$$[R \downarrow X] \uparrow Y = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0.7 & 0.7 & 0.7 & 0.7 & 0.7 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$[R \downarrow Y] \uparrow X = \begin{bmatrix} 0.9 & 0.5 & 1 & 0.4 & 1 \\ 0.9 & 0.5 & 1 & 0.4 & 1 \\ 0.9 & 0.5 & 1 & 0.4 & 1 \end{bmatrix}$$

min(1,0.9)

$$R_X \times R_Y = \begin{bmatrix} 0.9 & 0.5 & 1 & 0.4 & 1 \\ 0.7 & 0.5 & 0.7 & 0.4 & 0.7 \\ 0.9 & 0.5 & 1 & 0.4 & 1 \end{bmatrix}$$

INVERSE RELATION :

For a fuzzy relation $R(x,y)$ the inverse $R^{-1}(x,y)$ is defined by its membership funct given by

$$\mu_{R^{-1}}(y,x) \triangleq \mu_R(x,y) \text{ for all } (x,y) \in (X \times Y)$$

VARIOUS TYPES OF BINARY FUZZY RELATIONS : (10M)

→ The important types of binary fuzzy relations are classified on the basis of its characteristic property such as reflexivity, symmetric & transitivity.

→ A fuzzy relation $R(X,X)$ is reflexive if

$$\mu_R(x,x) = 1 \quad \forall\ x \in X$$

→ If the above equation is not satisfied for all $x \in X$ then the relation is called anti reflexive

→ If it is not satisfied for some $x \in X$ then the relation is called irreflexive

→ A fuzzy relation $R(X,X)$ is symmetric if

$$\mu_R(x, y) = \mu_R(y,x) \quad \text{for } x,y \in X$$

- If it is not satisfied for all the members of the relation then it is called anti symmetry

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If it is not satisfied for some $x, y \in X$ then it is called asymmetric

- A fuzzy relation $R(X, X)$ is transitive (max-min transitive)

if

$$\mu_R(x, z) \geq \max_{y \in Y} \min \left[\mu_R(x, y), \mu_R(y, z) \right] \text{ for all } (x, z) \in X^2$$

replace w t-norm $(\circ)$

→ If it does not hold good for all $(x, z) \in X^2$ then the relation $R(X, X)$ is anti transitive.

→ If it is not true for some members of X then relation $R(X, X)$ is called non transitive.

→ The max-min transitive can be generalized to alternate transitives by replacing min operator with any t-norm operator.

- Using algebraic product, the max productive transitive is defined

$$\mu_R(x, z) \geq \max_{y \in Y} \left[\mu_R(x, y) \cdot \mu_R(y, z) \right] \text{ for all } (x, z) \in X^2$$

In short

Reflexivity : $\mu_R(x_i, x_i) = 1$

Symmetry : $\mu_R(x_i, x_j) = \mu_R(x_j, x_i)$

Transitivity : $\mu_R(x_i, x_j) = \lambda_1$ &

$\mu_R(x_j, x_K) = \lambda_2$ then

$\mu_R(x_i, x_K) = \lambda$ where $\lambda \geq \min[\lambda_1, \lambda_2]$

OBLEM :

For the following fuzzy relations identify which propertie

hold good for binary fuzzy relation

reflexive

$$R_a = \begin{bmatrix} 1 & 0.8 & 0.3 \\ 0.3 & 1 & 0.6 \\ 0.4 & 0 & 1 \end{bmatrix}$$

antireflexive

$$R_b = \begin{bmatrix} 0.3 & 1 & 0.9 \\ 0 & 0.7 & 0.2 \\ 0.5 & 0 & 0.3 \end{bmatrix}$$

symmetric

$$R_c = \begin{bmatrix} 1 & 0.5 & 0.7 \\ 0.5 & 0.3 & 0.1 \\ 0.7 & 0.1 & 0 \end{bmatrix}$$

asymmetric

$$R_d = \begin{bmatrix} 1 & \overset{1,2}{\underline{0}} & \overset{13}{0.6} \\ \overset{(2,1)}{\underline{0}} & \overset{22}{0.3} & \overset{23}{0.8} \\ \overset{31}{0.5} & \overset{32}{0.7} & \overset{33}{0.5} \end{bmatrix}$$

$$R_e = \begin{bmatrix} 1 & \overset{1,2}{0} & \overset{1,3}{0.6} \\ \overset{2,1}{0.1} & 3 & \overset{2,3}{0.8} \\ \overset{3,1}{0.5} & \overset{3,2}{0} & 0.5 \end{bmatrix}$$

antisymmetric

$$R_f = \begin{bmatrix} 0.1 & 0.5 & 0.7 \\ 0 & 1 & 0.2 \\ 0 & 0.3 & 0.2 \end{bmatrix}$$

transitive

Sol:

$$R_a = \begin{bmatrix} 1 & 0.8 & 0.3 \\ 0.3 & 1 & 0.6 \\ 0.4 & 0 & 1 \end{bmatrix}$$

$x_{11} = 1,\ x_{22} = 1,\ x_{33} = 1$

Reflexive

$$R_b = \begin{bmatrix} 0.3 & 1 & 0.9 \\ 0 & 0.4 & 0.2 \\ 0.5 & 0 & 0.3 \end{bmatrix}$$

anti reflexive

$$R_c = \begin{bmatrix} 1 & 0.5 & 0.7 \\ 0.5 & 0.3 & 0.1 \\ 0.7 & 0.1 & 0 \end{bmatrix}$$

$x_{12} = x_{21} \quad x_{13} = x_{31}$

$x_{23} = x_{32}$ Symmetric.

$$R_d = \begin{bmatrix} 1 & 0 & 0.6 \\ 0 & 0.3 & 0.8 \\ 0.5 & 0.7 & 0.5 \end{bmatrix}$$

Assymmetric

$$R_e = \begin{bmatrix} 1 & 0 & 0.6 \\ 0.1 & 3 & 0.8 \\ 0.5 & 0 & 0.5 \end{bmatrix}$$

antisymmetric

$$R_f = \begin{bmatrix} 0.1 & 0.5 & 0.7 \\ 0 & 1 & 0.2 \\ 0 & 0.3 & 0.2 \end{bmatrix}$$

$\rightarrow$

SIMILARITY RELATIONS (OR) EQUIVALENCE RELATION:

- A binary fuzzy relation which is reflexive, symmetry & transitive is known as similarity relations.

The similarity relation is represented in its resolution form i.e. $R = \bigcup_{\alpha \in \Lambda_R} \alpha R_\alpha$ each α-cut R_α is an equivalence relation representing presence of similarity of elements to a degree α

For this equivalence relation R_α there is a partition on X, πR_α. Therefore, each similarity relation is associated with the set $\pi(R) = \{\pi(R_\alpha) \mid \alpha \in \Lambda_R\}$ (forall)

Problem (imp 10m) internal 1

Let $X = \{x_1, x_2, x_3, x_4, x_5\}$ consider the similarity relation

$$R(x,x) = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{bmatrix} 1 & \overset{1,2}{0.7} & \overset{1,3}{0.3} & 0.6 & 0.7 \\ 0.7 & 1 & 0.3 & 0.6 & 0.9 \\ 0.3 & 0.3 & 1 & 0.3 & 0.3 \\ 0.6 & 0.6 & 0.3 & 1 & 0.6 \\ 0.7 & 0.9 & 0.3 & 0.6 & 1 \end{bmatrix} \end{array}$$

Q:- Draw the partition tree for above similarity relation.

Level set $\Lambda_R = \{0.3, 0.6, 0.7, 0.9, 1\}$

$0.3 \leq \rightarrow 1$

$$R_{0.3} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$R_{0.6} = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{bmatrix}$$

$$R_{0.7} = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$R_{0.9} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$R_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The equivalence classes for these equivalence relation are

having 1

$\pi(R_1) = \{\{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}, \{x_5\}\}$

$\pi(R_{0.9}) = \{\{x_1\}, \{x_2, x_5\}, \{x_3\}, \{x_4\}\}$ repeating dont write

$\pi(R_{0.7}) = \{\{x_1, x_2, x_5\}, \{x_3\}, \{x_4\}\}$

$\pi(R_{0.6}) = \{\{x_1, x_2, x_4, x_5\}, \{x_3\}\}$

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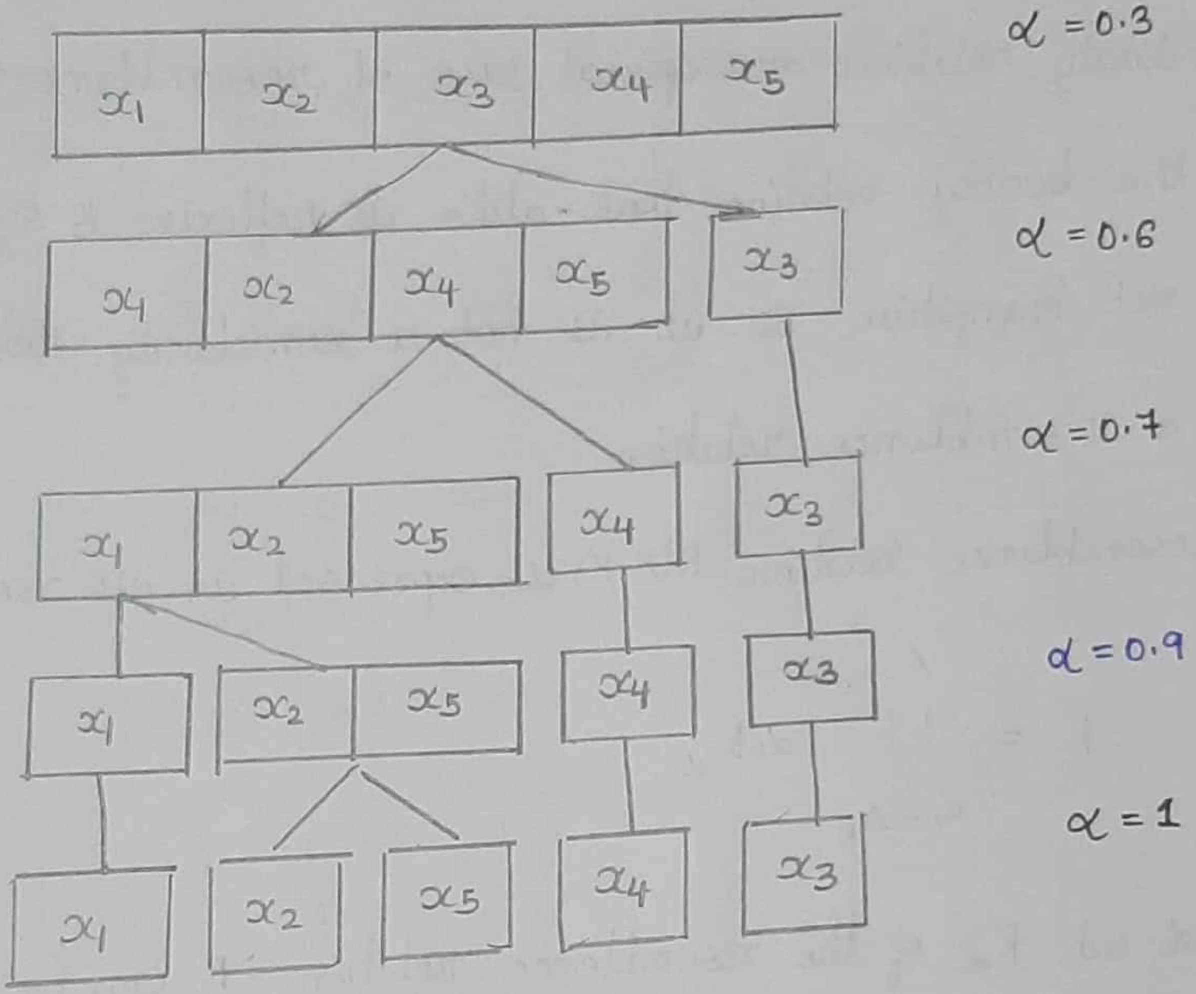
$\pi(R_{0.3}) = \{\{x_1, x_2, x_3, x_4, x_5\}\}$

The collection of these partition $\pi(R) = \{\pi(R_1), \pi(R_{0.9}), \pi(R_{0.7}), \pi(R_{0.6}), \pi(R_{0.3})\}$ can be shown in a tree like structure called Partition tree as shown below



RESEMBLENCE RELATION (OR) TOLERANCE RELATION :

- A binary fuzzy relation which is reflexive & symmetric is called resemblance relation
- Similarity relations are special case of resemblance relation

eg - the binary relation look-alike is reflexive & symmetric but not transitive. So it is not a similarity relation but a resemblance relation.

- A resemblance relation $R(X,X)$ is expressed in its resolution form

$$R = \bigcup_{\alpha \in \Lambda_R} \alpha R_\alpha$$

the α-cut R_α of the resemblence relation R can be defined as A_α is a maximal subset of X such that

$$\mu_R(x,y) \geq \alpha \text{ for all } x, y \in A_\alpha$$

- The family consisting of all resemblence classes A_α is called ~~α power~~ α-cover of X w.r.t R_α

- ∴ for each α-cut R_α there is an α-cover of X that corresponds to partition $\pi(c)$ of an equivalence relation.
- By this, the complete α-cover tree consisting of all α-covers, $\alpha \in \Lambda_R$ is drawn

Problem

- Let $X = \{x_1, x_2, x_3, x_4, x_5\}$ & resemblence relation

$$R(X, X) = \begin{array}{c|ccccc} & x_1 & x_2 & x_3 & x_4 & x_5 \\ \hline x_1 & 1 & 0.6 & 0.3 & 0.3 & 0.7 \\ x_2 & 0.6 & 1 & 0.3 & 0.3 & 0.9 \\ x_3 & 0.3 & 0.3 & 1 & 0.3 & 0.3 \\ x_4 & 0.3 & 0.3 & 0.3 & 1 & 0.7 \\ x_5 & 0.7 & 0.9 & 0.3 & 0.7 & 1 \end{array}$$

$$R_{0.3} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$R_{0.6} = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{bmatrix}$$

$$R_{0.7} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{bmatrix} \qquad R_{0.9} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$R_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\pi(R_1) = \{\{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}, \{x_5\}\}$$

$$\pi(R_{0.9}) = \{\{x_1\}, \{x_2, x_5\}, \{x_3\}, \{x_4\}\}$$

$$\pi(R_{0.7}) = \{\{x_1, x_5\}, \{x_2, x_5\}, \{x_3\}, \{x_4, x_5\}, \{x_1, x_2, x_4,$$

$$\pi(R_{0.6}) = \{\{x_1, x_2, x_5\}, \{x_3\}, \{x_4, x_5\}\}$$

$\{x_1, x_2, x_4, x_5\}$

repeated / can write

$$\pi(R_{0.3}) = \{\{x_1, x_2, x_3, x_4, x_5\}\}$$

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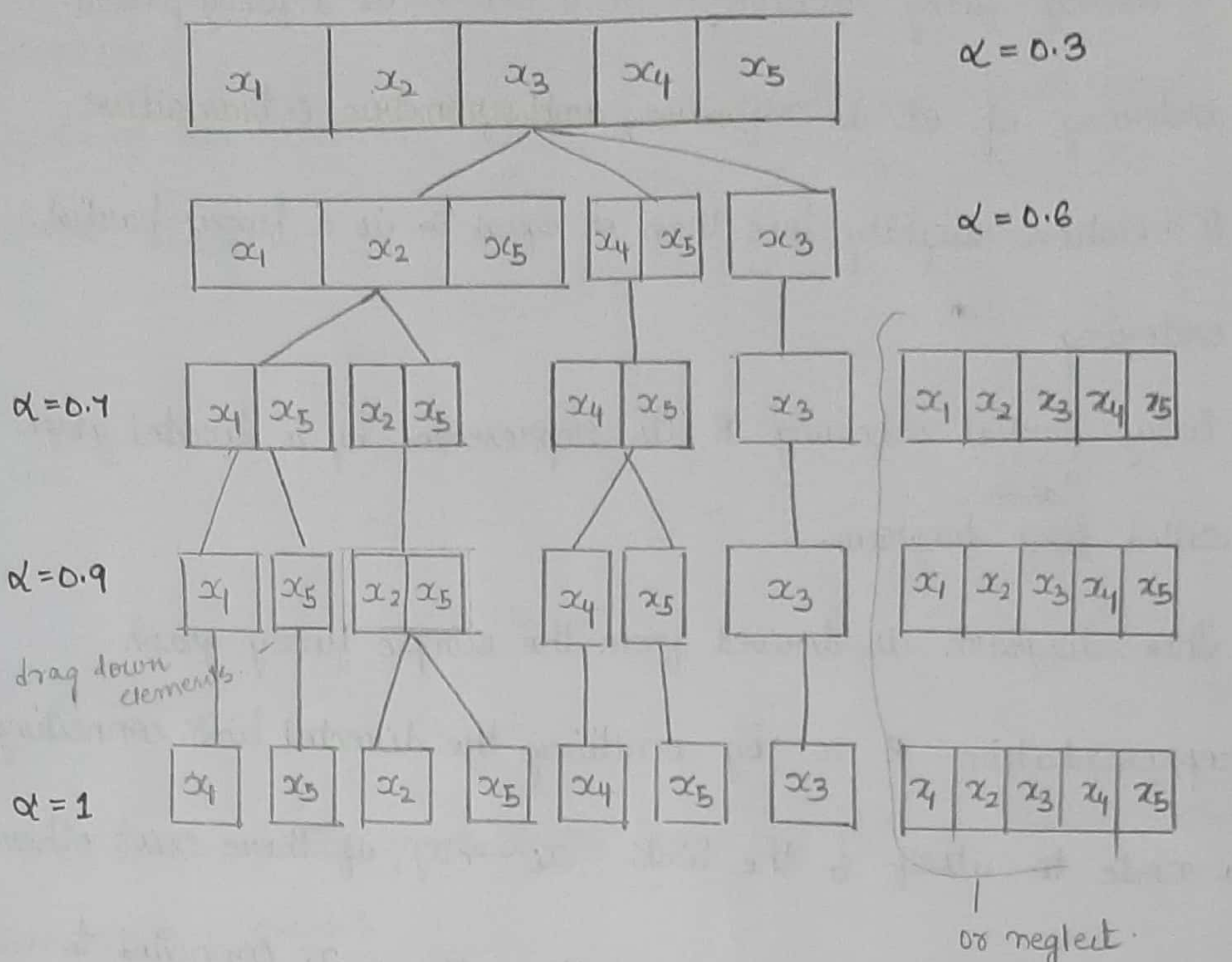
α-COVER TREE :



(covering all elements which were neglected in tree partion elements are taking down along)

FUZZY PARTIAL RELATIONS FOR ORDERING

- A binary fuzzy relation R on a set X is a fuzzy partial ordering if it is reflexive, antisymmetric & transitive
- A relation slightly less than or equal to is a fuzzy partial ordering
- Every partial ordering R is represented by a directed graph called ~~[illegible]~~ Hasse diagram.

This diagram is derived from the simple fuzzy graph representation R r by omitting the directed link connecting a node to itself & the link $x_i \rightarrow x_j$ if there exist other nodes $x_{K1}, x_{K2}, --- x_{Kn}$ such than x_{K1} & x_i connected to x_{K1} connected to x_{Ki} so on connected to x_{Kn} connected to x_j

$$x_i \rightarrow x_{K1} \rightarrow x_{Ki} \rightarrow --- \rightarrow x_{Kn} \rightarrow x_j$$

where n can be equal to 1

∴ in a Hasse diagram if there is a link from x_i to x_j, x_j is called an immediate successor of

x_i & x_i is called an immediate predecessor of x_j

$(x_i \rightarrow x_j)$

PROBLEM :

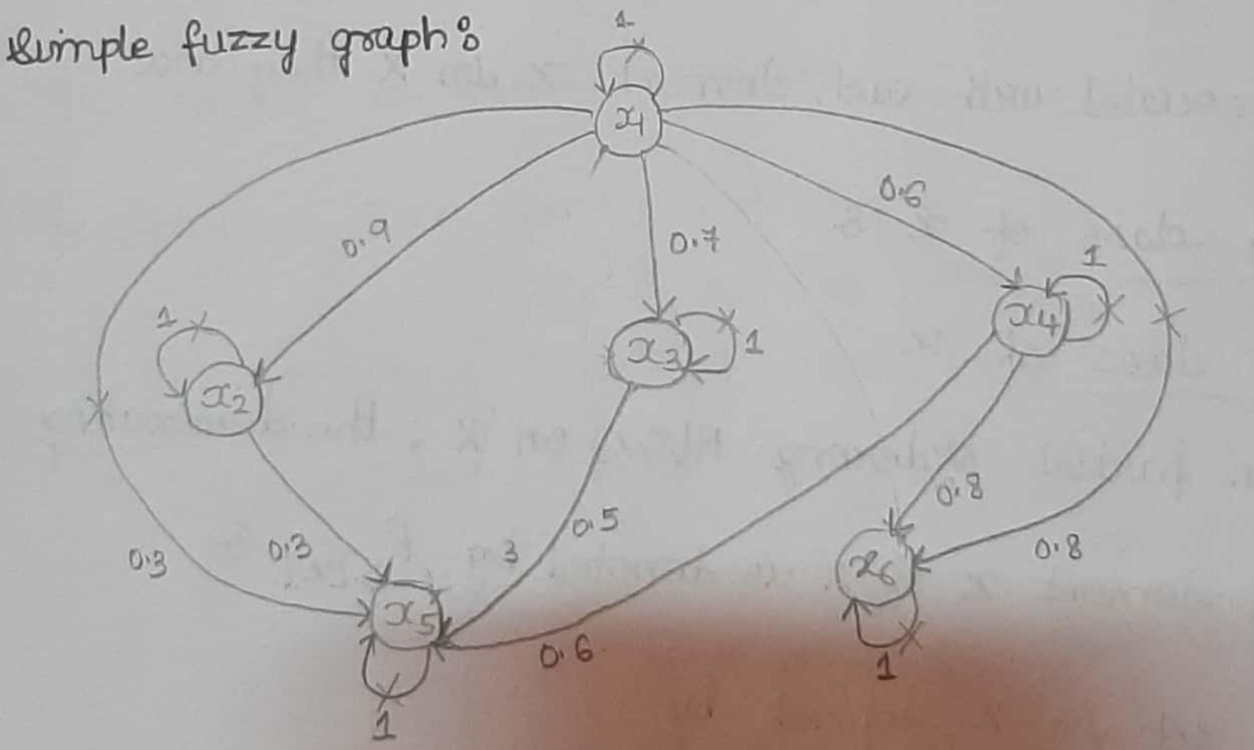
- Consider the following fuzzy partial ordering $R(X,X)$ on X

where $R(X,X) =$

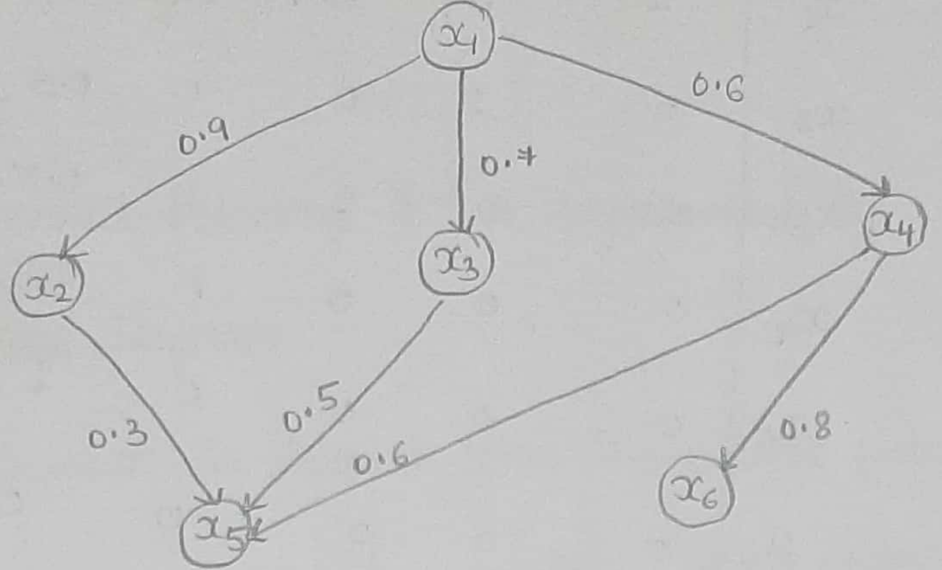
$$
\begin{array}{c|cccccc}
 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\
\hline
x_1 & 1 & 0.9 & 0.7 & 0.6 & 0.3 & 0.8 \\
x_2 & 0 & 1 & 0 & 0 & 0.3 & 0 \\
x_3 & 0 & 0 & 1 & 0 & 0.5 & 0 \\
x_4 & 0 & 0 & 0 & 1 & 0.6 & \cancel{0.8} \\
x_5 & 0 & 0 & 0 & 0 & 1 & 0 \\
x_6 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}
$$

- The simple fuzzy graph & the hasse dig of R is shown below.

simple fuzzy graph :



Hasse diagram:



shortans

DOMINATING, DOMINATED CLASS-

→ For a given partial ordering $R(X,X)$, two important fuzzy sets are associated with each element x in X they are

1. Dominating class of x &
2. Dominated class of x

– For a given partial ordering $R(X,X)$ on X, the dominating class of an element x of X is denoted by $R_{\geq[x]}$ & is a fuzzy set in X defined by

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$$\mu_{R_{\geq[x]}}(y) = \mu_R(x,y) \; ; \; y \in Y$$

- Similarly the dominated class for an element x of X is denoted by $R_{\leq[x]}$ & is a fuzzy set in X defined by

$$\mu_{R_{\leq[x]}}(y) = \mu_R(y,x) \; ; \; y \in Y$$

- Element 'x' is called a maximal or undominated element if

$$\mu_R(x,y) = 0 \text{ for all } y \in X \text{ \& } y \neq x$$

- Element 'x' is called minimal or undominating element if

$$\mu_R(y,x) = 0 \text{ for all } y \in X \text{ \& } y \neq x$$

less element in column → minimal

- Eg: Consider fuzzy partial ordering

$$R(X,X) = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{array} \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array} \\ \begin{bmatrix} 1 & 0.9 & 0.4 & 0.6 & 0.3 & 0.8 \\ 0 & 1 & 0 & 0 & 0.3 & 0 \\ 0 & 0 & 1 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 1 & 0.6 & 0.8 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

- x_5, x_6 are maximal element & x_1 is minimal element

BOUND :

- Let R be a fuzzy partial ordering on X and let A be a subset of X. The fuzzy upper bound of A is a fuzzy set U(A) defined by

$$U(A) = \bigcap_{x \in A} R_{\geq [x]}$$

when $\cap$ is fuzzy intersection operation

- The least upper bound of set A is the unique element in U(A) such that $\mu_{U(A)}(x) > 0$ & $\mu_R(x,y) > 0$ for all elements y in the support of U(A)

$$y \rightarrow U(A)$$

- The least upper bound of A is the smallest element of U(A).

- The fuzzy lower bound of A is the fuzzy set L(A) defined by $L(A) = \bigcap_{x \in A} R_{\leq [x]}$

→ The greatest lower bound of set A is the unique element in $L(A)$ such that $\mu_{L(A)}(x) > 0$ & $\mu_R(y, x) > 0$ for all elements y in the support of $L(A)$

- The greatest lower bound of A is the largest element of $L(A)$

$$
\begin{array}{cc}
 & \overbrace{\hphantom{x_2 \quad x_3 \quad x_4}}^{\text{lower}} \quad \text{minimal} \\
\end{array}
$$

$$
R(x, x) = \begin{array}{c|cccccc}
 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\
\hline
x_1 & 1 & 0.9 & 0.7 & 0.6 & 0.3 & 0.8 \\
x_2 & 0 & 1 & 0 & 0 & 0.3 & 0 \\
x_3 & 0 & 0 & 1 & 0 & 0.5 & 0 \\
x_4 & 0 & 0 & 0 & 1 & 0.6 & 0.8 \\
x_5 & 0 & 0 & 0 & 0 & 1 & 0 \\
x_6 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}
$$

upper (x_2, x_3, x_4)

minimal

Consider following fuzzy ordering

Let $A = \{x_2, x_3, x_4\}$ Find the fuzzy upper bound of A & fuzzy lower bound of A.

Sol: Fuzzy upper bound of A is given by

$$U(A) = \bigcap_{x \in A} R_{\geq [x]}$$

$\therefore U(A) = R_{\geq[x_2]} \cap R_{\geq[x_3]} \cap R_{\geq[x_4]}$

$$= \left(\frac{1}{x_2} + \frac{0.3}{x_5}\right) \cap \left(\frac{1}{x_3} + \frac{0.5}{x_5}\right) \cap \left(\frac{1}{x_4} + \frac{0.6}{x_5} + \frac{0.8}{x_6}\right)$$

$$= \quad \frac{0.3}{x_5} \qquad \frac{0.5}{x_5} \qquad \frac{0.6}{x_5}$$

(minimal)

$$= \frac{0.3}{x_5}$$

The fuzzy lower bound of A is fuzzy set L(A) given by

$$L(A) = \bigcap_{x \in A} R_{\leq[x]}$$

$\therefore L(A) = R_{\leq[x_2]} \cap R_{\leq[x_3]} \cap R_{\leq[x_4]}$

$$= \cancel{\left(\frac{1}{x_2} + \frac{0.3}{x_5}\right) \cap \left(\frac{1}{x_3}\right.}$$

consider column element

$$= \left(\frac{0.9}{x_1} + \frac{1}{x_2}\right) \cap \left(\frac{0.7}{x_1} + \frac{1}{x_3}\right) \cap \left(\frac{0.6}{x_1} + \frac{1}{x_4}\right)$$

$$= \quad 0.9/x_1 \qquad \frac{0.7}{x_1} \qquad \frac{0.6}{x_1}$$

(minimal)

$$= \frac{0.6}{x_1}$$